

## A nice theorem on monotonic predicate sequences

A predicate sequence  $P_i$  ( $i \geq 0$ ) such that  $\forall$ -denoting universal quantification over all states by square brackets— $(\underline{A}i: i \geq 0: [P_i \Rightarrow P_{i+1}])$  is called weakening; a predicate sequence  $P_i$  ( $i \geq 0$ ) such that  $(\underline{A}i: i \geq 0: [P_{i+1} \Rightarrow P_i])$  is called strengthening. A predicate sequence  $P_i$  ( $i \geq 0$ ) that is weakening or strengthening is called monotonic.

Theorem For any monotonic predicate sequence  $P_i$  ( $i \geq 0$ ) we have

$$[(\underline{A}i: i \geq 0: (\underline{E}j: j \geq i: P_j))] = (\underline{E}i: i \geq 0: (\underline{A}j: j \geq i: P_j)) .$$

Proof. Let  $P_i$  ( $i \geq 0$ ) be a weakening predicate sequence; we then have for all natural  $i$

$$(0) \quad [P_i = (\underline{A}j: j \geq i: P_j)]$$

$$(1) \quad [(\underline{E}j: j \geq i: P_j) = (\underline{E}j: j \geq 0: P_j)] .$$

For arbitrary  $Z$  we observe

$$\begin{aligned} & [Z = (\underline{A}i: i \geq 0: (\underline{E}j: j \geq i: P_j))] \\ &= \{(1)\} \\ & [Z = (\underline{A}i: i \geq 0: (\underline{E}j: j \geq 0: P_j))] \\ &= \{\text{predicate calculus}\} \\ & [Z = (\underline{E}j: j \geq 0: P_j)] \\ &= \{\text{renaming the dummy}\} \\ & [Z = (\underline{E}i: i \geq 0: P_i)] \\ &= \{(0)\} \\ & [Z = (\underline{E}i: i \geq 0: (\underline{A}j: j \geq i: P_j))] \end{aligned}$$

which proves the theorem for weakening  $P_i$  ( $i \geq 0$ ).

Let  $P_i$  ( $i \geq 0$ ) be a strengthening predicate sequence; then

the sequence of predicates  $\neg P_i$  is weakening and we conclude from the above

$$[(\underline{A}i: i \geq 0: (\underline{E}j: j \geq i: \neg P_j))] = (\underline{E}i: i \geq 0: (\underline{A}j: j \geq i: \neg P_j)) .$$

Negating both sides and applying de Morgan yields

$$[(\underline{E}i: i \geq 0: (\underline{A}j: j \geq i: P_j))] = (\underline{A}i: i \geq 0: (\underline{E}j: j \geq i: P_j)) .$$

(End of Proof.)

As we see from the proof, the two equal expressions mentioned in the Theorem are for a weakening predicate sequence  $P_i$  ( $i \geq 0$ ) most simply expressed as  $(\underline{E}i: i \geq 0: P_i)$ ; for a strengthening predicate sequence  $P_i$  ( $i \geq 0$ ) they are most simply expressed as  $(\underline{A}i: i \geq 0: P_i)$ . These are the forms in which these limits are best known.

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C.S. Schotten and I discovered this theorem while working on EWD816. We were slightly amazed that in its full generality the Theorem was new for us, the more so because monotonic predicate sequences occur so frequently and its proof is so simple. (The Theorem, though nice and in a way striking, is definitely not deep.) Hence this note.

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3 April 1982  
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