The maximum length of a segment satisfying a monotonic predicate

For a given sequence $f(i: 0 \le i < N)$, $f(i: x \le i < y)$ with $0 \le x \le y \le N$ is called "a segment of length $y-x$".

Let, with $0 \le x \le y \le N$, $B\,x\,y$ be some predicate on segment $f(i: x \le i < y)$ such that

$$\underline{A}(x,h,k,y: 0 \le x \le h \le k \le y \le N: B\,h\,k \vee \neg B\,x\,y) \quad ;$$

such a predicate is called "monotonic". We know many examples of monotonic predicates, such as:

- all elements positive
- all elements equal
- ascending
- not containing adjacent, non-empty, equal subsegments
- having its first differences of alternating signs.

For a B that holds for any empty segment we shall derive a program establishing R given by

$$R: \quad c = \underline{MAX}(x,y: 0 \le x \le y \le N \wedge B\,x\,y: y-x) \quad .$$

To begin with we approach the problem in the standard way by introducing a variable n satisfying $P0$ given by

$$P0: \quad c = \underline{MAX}(x,y: 0 \le x \le y \le n \wedge B\,x\,y: y-x) \wedge 0 \le n \le N \quad ,$$

which yields a program of the structure

```
|[ n: int ; c, n := 0,0 {invariant P0}
; do n ≠ N →
      "increase n by 1 under invariance of P0"
  od
]|    .
```

From $P0 \wedge n \neq N$ we conclude

$$\underline{\text{MAX}}(x,y: 0 \leq x \leq y \leq n+1 \wedge B\, x\, y : y-x) =$$
$$\underline{\text{MAX}}(x: 0 \leq x \leq n+1 \wedge B\, x\, (n+1) : n+1-x)\ \underline{\text{max}}\ c \quad ;$$

for the sake of convenience we rewrite the last line as

$$(n+1 - \underline{\text{MIN}}(x: 0 \leq x \leq n+1 \wedge B\, x\, (n+1) : x))\ \underline{\text{max}}\ c \quad ,$$

which suggests the introduction of a variable h satisfying $P1$, given by

$$P1: \quad h = \underline{\text{MIN}}(x: 0 \leq x \leq n \wedge B\, x\, n : x) \quad .$$

This yields a program of the structure

```
|[ n,h: int ; c,n,h := 0,0,0 {invariant P0 ∧ P1}
; do n ≠ N →
      "establish P1(n+1/n)"
     ; c := (n+1-h) max c {P0(n+1/n)}
     ; n:= n+1 {P0 ∧ P1}
  od
]|    .
```

Without exploiting any property of B (beyond the fact that it holds for the empty segment), the Linear Search Theorem tells us that there is only one way of establishing $P1(n+1/n)$, viz.

$$h:=0\ ;\ \underline{do}\ \neg B\ h\ (n+1) \rightarrow h:=h+1\ \underline{od} \quad ,$$

which —disregarding the evaluations of B— gives in general rise to a quadratic algorithm.

From the monotonicity of B, however, we can conclude that the solution of the equation $h: P1$ is at most the solution of $h: P1(n+1/n)$; hence, establishing $P1(n+1/n)$ can be implemented by

$$\underline{do}\ \neg B\ h\ (n+1) \rightarrow h:=h+1\ \underline{od} \quad ,$$

which —again disregarding the evaluations of B— gives rise to a linear algorithm.

Note. From the above analysis follows that the monotonicity requirement on B is stronger than necessary: a "one-sided" monotonicity

$$\underline{A}(x, k, y : 0 \leq x \leq k \leq y \leq N :\ B\,x\,k\ \vee \neg B\,x\,y)$$

would have sufficed. An example of such a B is

$$B\ x\ y \equiv \underline{A}(j : x \leq j < y :\ f x \leq f j) \quad .$$

(End of Note.)

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Three remarks are in order. We have postulated that B holds for the empty segment because —see R— we did not care to define $\underline{MAX}$ over an empty bag. If B holds for any one-element segment, it is often convenient to deal with $N=0$ separately; for $N>0$, the repetition can then be initialized with $n=1$ and has $h<n$ as a further invariant.

Secondly, the analytical structure of $\mathcal{B}$ is, thanks to some transitivity, often such that the nett effect of

$$\underline{\text{do}}\ \neg\, \mathcal{B}\ h\ (n+1) \rightarrow h := h+1\ \underline{\text{od}}$$

can be captured by a modest alternative statement, say of the form

$$\underline{\text{if}}\ \ldots \rightarrow \text{skip}\ [\!]\ \ldots \rightarrow h := n\ \underline{\text{fi}}\quad .$$

Thirdly, the assignment statement

$$c := (n+1-h)\ \underline{\text{max}}\ c$$

is equivalent to a skip in the case $n+1-h \leq c$, a situation implied by $N-h \leq c$. Once established, however, $N-h \leq c$ is an invariant of the repetition; hence we can strengthen the guard by its negation $h+c < N$. But since $n \leq h+c$ is a further invariant of the repetition $n \neq N \wedge h+c < N$ can be simplified to just $h+c < N$.

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*

By way of illustration we give the solution for

$$\mathcal{B}\ x\ y\ \equiv \underline{A}(j : x \leq j < y : f\,x \leq f\,j) \quad .$$

```
if N = 0 → c := 0
[] N > 0 → |[ n, h: int ; c, n, h := 1, 1, 0
            ; do h+c < N →
                 if f(n) ≥ f(h) → skip [] f(n) < f(h) → h := n fi
               ; n := n+1 ; c := (n-h) max c
              od
           ]|
fi .
```

The above $\mathcal{B}$ is one of one-sided monotonicity. Had we chosen

$$\mathcal{B}\ x\ y \equiv \underline{A}(j : x \leq j < y : f x = f j)$$

we would have posed The Plateau Problem (see [0], p. 203, which deals with the special case that the given sequence is ordered). Its solution is obtained by replacing the inner alternative statement in the above by

$$\underline{\text{if}}\ f(n) = f(h) \rightarrow \text{skip}\ [\!]\ f(n) \neq f(h) \rightarrow h := n\ \underline{\text{fi}} \quad .$$

[0] Gries, David; "The Science of Programming", Springer-Verlag, New York-Heidelberg-Berlin, 1981 .

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