Total-order junctivity

Continuity –see [0], p.87– is a special case of total-order junctivity. A set S of predicate transformers is totally ordered iff

$$\langle \forall x,y : x \in S \wedge y \in S : [x \Rightarrow y] \vee [x \Leftarrow y] \rangle \quad ,$$

and a predicate transformer is called "total-order junctive" iff it is junctive over totally ordered sets of predicates. We leave to the reader the proof of –see [0], p.84 –

$$S \text{ is totally ordered } \equiv S^* \text{ is totally ordered} \quad ,$$

and also of "any total-order junctive predicate transformer is monotonic". We now generalize Theorem (6,43) of [0] –see p.99– to

<u>Theorem</u> Disjunction preserves total-order conjunctivity, i.e. let predicate transformer f be given in terms of the total-order conjunctive predicate transformers g and h by

$$[f.x \equiv g.x \vee h.x] \quad \text{for all } x \ ;$$

then f is total-order conjunctive.

<u>Proof</u> In this proof, x and y range over an arbitrary totally ordered set. Our proof obligation is then to show

$$[f.\langle \forall x :: x \rangle \equiv \langle \forall x :: f.x \rangle] \quad .$$

To this end we observe

$$
\begin{array}{ll}
 & f.\langle\forall x :: x\rangle \\
= & \{\text{def. of } f \text{ and renaming a dummy}\} \\
 & g.\langle\forall x :: x\rangle \ \vee\ h.\langle\forall y :: y\rangle \\
= & \{g \text{ and } h \text{ total-order conjunctive}\} \\
 & \langle\forall x :: g.x\rangle \ \vee\ \langle\forall y :: h.y\rangle \\
= & \{\text{distribute } \vee \text{ over } \forall \text{; unnesting}\} \\
 & \langle\forall x,y :: g.x \vee h.y\rangle \\
= & \{\text{range is totally ordered}\} \\
 & \langle\forall x,y : [x \Rightarrow y] : g.x \vee h.y\rangle \ \wedge \\
 & \langle\forall x,y : [x \Leftarrow y] : g.x \vee h.y\rangle \\
= & \{\text{nesting}\} \\
 & \langle\forall x :: \langle\forall y : [x \Rightarrow y] : g.x \vee h.y\rangle\rangle \ \wedge \\
 & \langle\forall y :: \langle\forall x : [x \Leftarrow y] : g.x \vee h.y\rangle\rangle \\
= & \{h \text{ monotonic; } g \text{ monotonic}\} \\
 & \langle\forall x :: g.x \vee h.x\rangle \ \wedge\ \langle\forall y :: g.y \vee h.y\rangle \\
= & \{\text{def. of } f\} \\
 & \langle\forall x :: f.x\rangle
\end{array}
$$

(End of Proof.)

The analog of Theorem (5,116) of [0] — see p. 76-77 — is:

With f monotonic in both arguments and x and y ranging over some totally ordered set

$$[\langle\forall x,y :: f.x.y\rangle \equiv \langle\forall x :: f.x.x\rangle] \quad ,$$

which we could have appealed to in the above proof. All this is more general, cleaner and simpler than in [0]. For the sake of completeness (and having just started a new page) we prove the last theorem by observing

$\langle\forall x,y:: f.x.y\rangle$
$=$ { range is totally ordered }
$\langle\forall x,y: [x\Rightarrow y]: f.x.y\rangle \wedge \langle\forall x,y: [x\Leftarrow y]: f.x.y\rangle$
$=$ { nesting }
$\langle\forall x:: \langle\forall y: [x\Rightarrow y]: f.x.y\rangle\rangle \wedge \langle\forall y:: \langle\forall x: [x\Leftarrow y]: f.x.y\rangle\rangle$
$=$ { f monotonic in both arguments }
$\langle\forall x:: f.x.x\rangle \wedge \langle\forall y:: f.y.y\rangle$
$=$ { pred. calc. }
$\langle\forall x:: f.x.x\rangle$.

All this has been triggered by a theorem in the thesis of Ernie Cohen.

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[0] Edsger W. Dijkstra and Carel S. Scholten, Predicate Calculus and Program Semantics, Springer-Verlag, New York - Berlin - Heidelberg, 1990