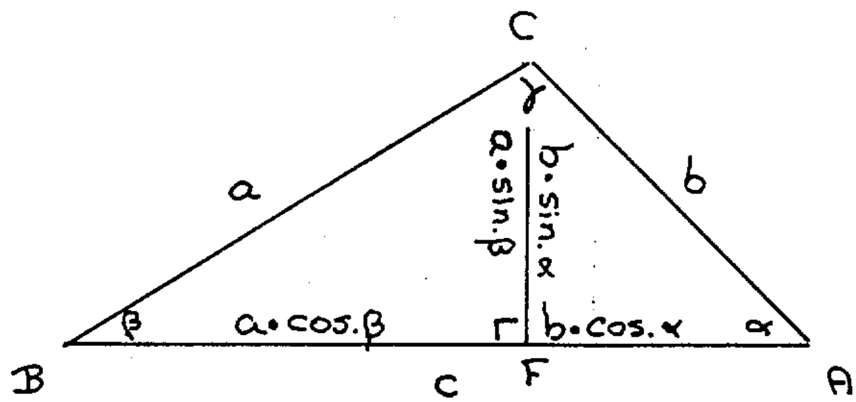
The formula for sin.(α+β)

We consider a triangle with sides a, b, c and opposite angles α, β, γ respectively:



We have added the altitude CF; the additional annotation follows from the definitions of the sine and cosine functions. We observe

$$
\begin{array}{ll}
 & true \\
\equiv & \{\text{the two annotations for CF}\} \\
 & a \cdot \sin.\beta = b \cdot \sin.\alpha \\
\equiv & \{\text{algebra}\} \\
 & a : b = \sin.\alpha : \sin.\beta \\
\equiv & \{\text{symmetry}\} \\
 & a : b : c = \sin.\alpha : \sin.\beta : \sin.\gamma \qquad (*)
\end{array}
$$

Next we observe

$$
\begin{array}{ll}
 & true \\
\equiv & \{\text{annotations for BF and FA}\}
\end{array}
$$

$$
\begin{array}{ll}
 & c = a \cdot \cos.\beta + b \cdot \cos.\alpha \qquad (\dagger) \\
\equiv & \{(*)\} \\
 & \sin.\gamma = \sin.\alpha \cdot \cos.\beta + \sin.\beta \cdot \cos.\alpha \\
\equiv & \{\alpha + \beta + \gamma = \pi\} \\
 & \sin.(\alpha+\beta) = \sin.\alpha \cdot \cos.\beta + \sin.\beta \cdot \cos.\alpha \qquad (**)
\end{array}
$$

and so we have proved the addition formula $(**)$ for the sine function for $0 \leq \alpha$, $0 \leq \beta$ and $\alpha + \beta \leq \pi$. (Note that F does not need to lie between A and B for $(\dagger)$ to be valid.)

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